A Probabilistic Model for Image Processing with Positivity Constraint and Spectral Density Control

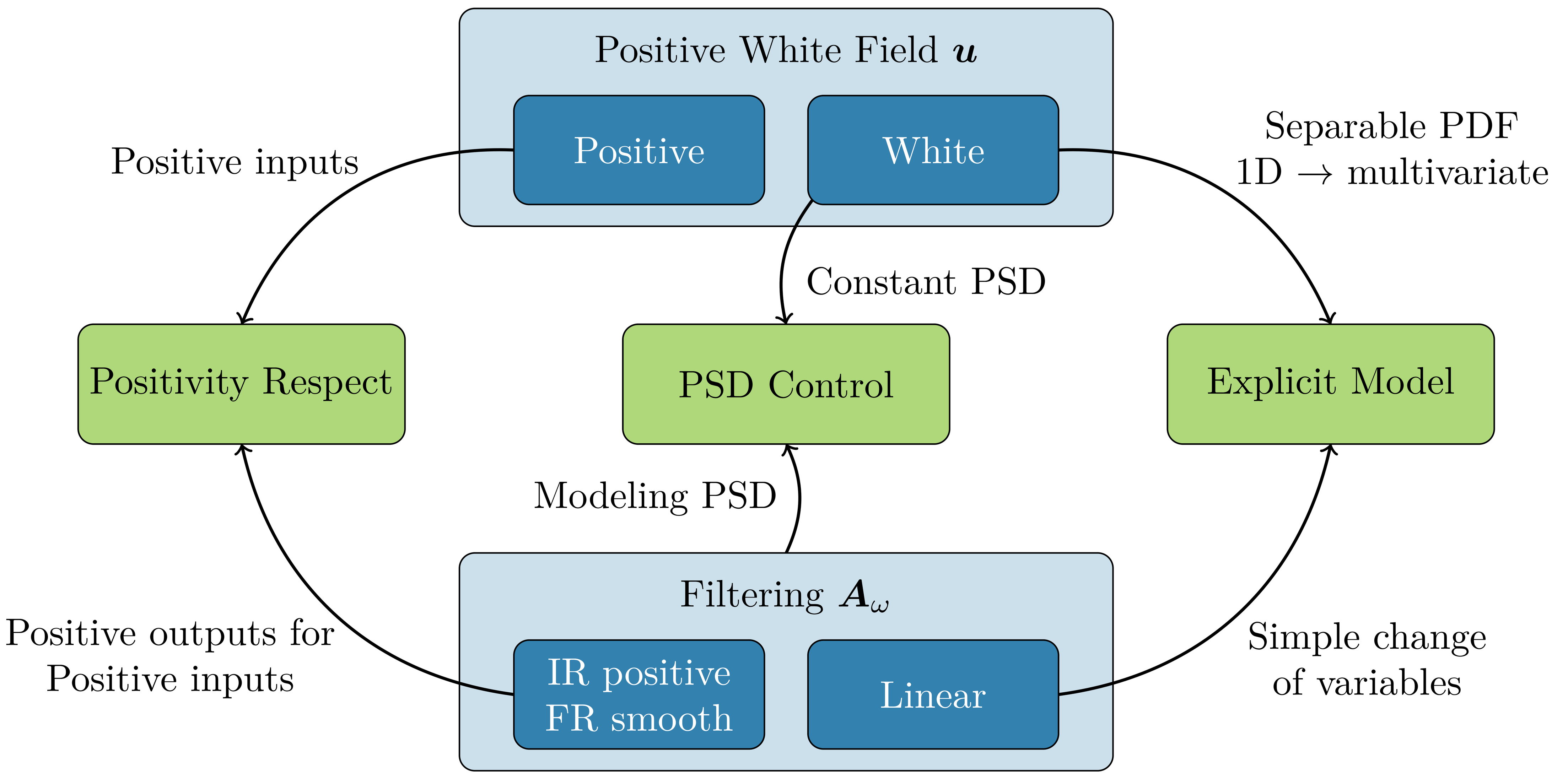
Pierre Minier[1], Jean-François Giovannelli[1] and François Orieux[2]

[1]IMS (Univ. Bordeaux – CNRS – BINP) and [2]L2S (Univ. Paris-Saclay – CNRS – CentraleSupélec)

Introduction

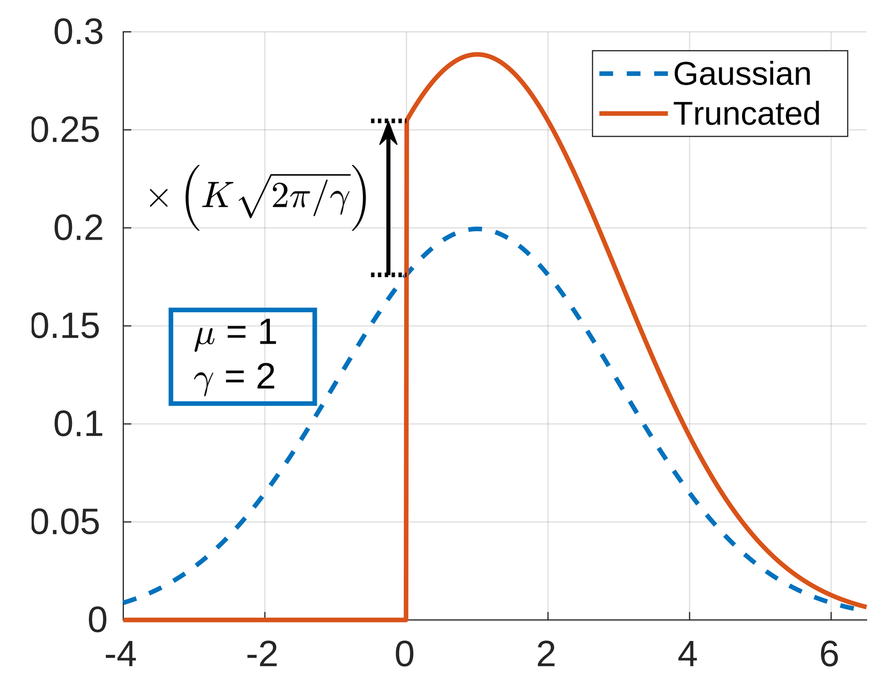
We propose a novel probabilistic image model that ensures **pixel positivity**, controls the **Power Spectral Density**, and provides an **explicit partition function**. This fills a gap in the literature and lays the groundwork for a fully self-supervised image deconvolution method with such properties.

Model Construction $\boldsymbol{x} = \boldsymbol{A}_\omega \boldsymbol{u}$



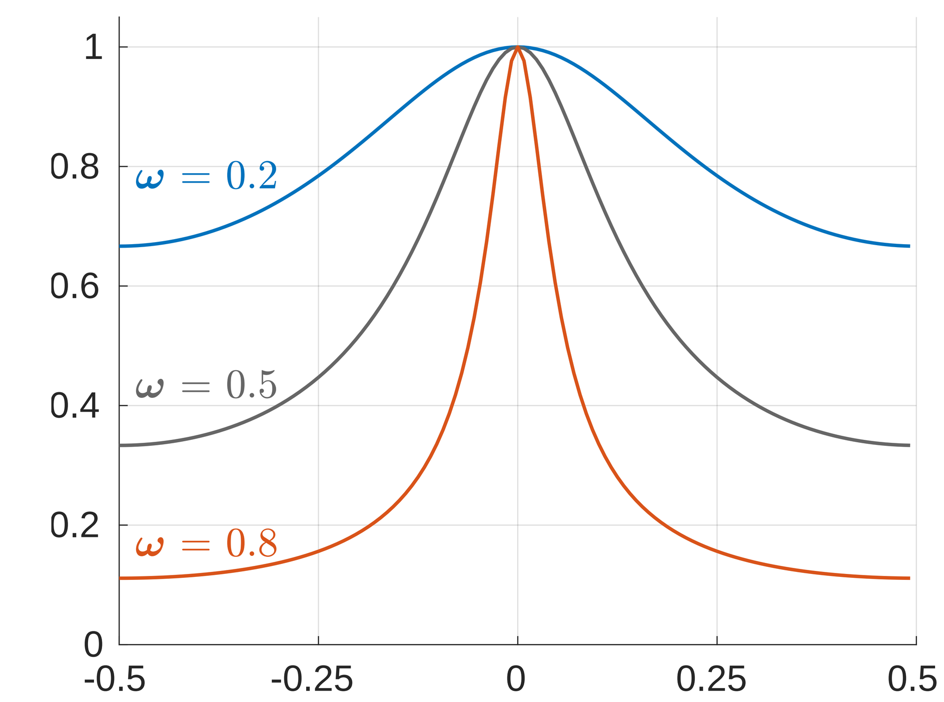
$$f(\boldsymbol{x} \mid \boldsymbol{\eta}) = |\det \boldsymbol{A}_\omega|^{-1} K^P \exp\left(-\frac{\gamma}{2}\|\boldsymbol{A}_\omega^{-1}\boldsymbol{x} - \mu\mathbb{1}\|^2\right) \mathbb{1}_+(\boldsymbol{A}_\omega^{-1}\boldsymbol{x}) \text{ with parameters } \boldsymbol{\eta} = [\mu, \gamma, \omega]$$

Positive White Field



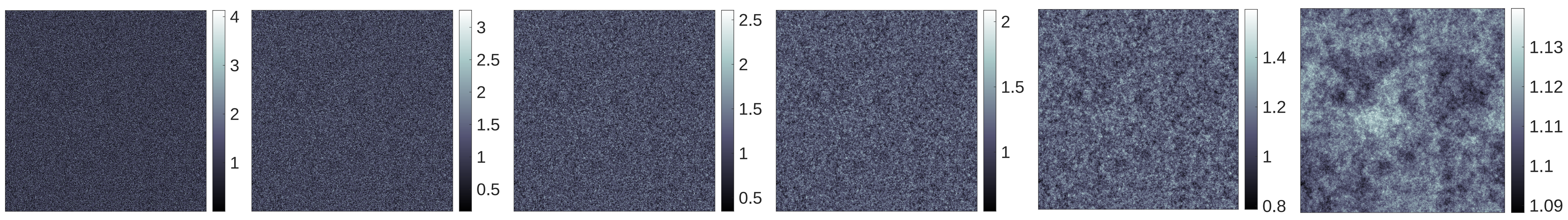
Truncated Gaussian.

Filtering



Cross section of the FR used.
AR(1) extended in 2D by circular symmetry.

Model Realisations



$\omega = 0$ | $\omega = 0.2$ | $\omega = 0.4$ | $\omega = 0.6$ | $\omega = 0.8$ | $\omega = 0.99$

Parameter Estimation

Model Density in the Fourier Domain

$\boldsymbol{A}_\omega$ is CBBC $\Rightarrow$ diagonalizable in the Fourier domain $\Rightarrow$ faster computation.

$$f(\boldsymbol{x} \mid \boldsymbol{\eta}) = |\det \boldsymbol{\Lambda}_\omega|^{-1} K^P \exp\left(-\frac{\gamma}{2}\sum_{p=0}^{P-1}\left|\frac{\mathring{x}_p}{\lambda_\omega(p)} - \mu\mathring{\mathbb{1}}_p\right|^2\right) \mathbb{1}_+(\boldsymbol{F}^{-1}\boldsymbol{\Lambda}_\omega^{-1}\mathring{\boldsymbol{x}})$$

- $\boldsymbol{F}$ the matrix of the discretised 2D Fourier transform.
- $\boldsymbol{\Lambda}_\omega = \mathrm{diag}[\lambda_\omega(p), p = 0\cdots(P-1)]$ and $\lambda_\omega(p)$ the discretised FR.
- $K = \sqrt{2\gamma/\pi}\,/\,[1 + \mathrm{erf}(\mu\sqrt{\gamma/2})]$ the partition of the truncated 1D gaussian.

Bayesian Estimation

The true parameters μ^*, γ^* and ω^* generate a sample $\boldsymbol{x}$ used for their estimation

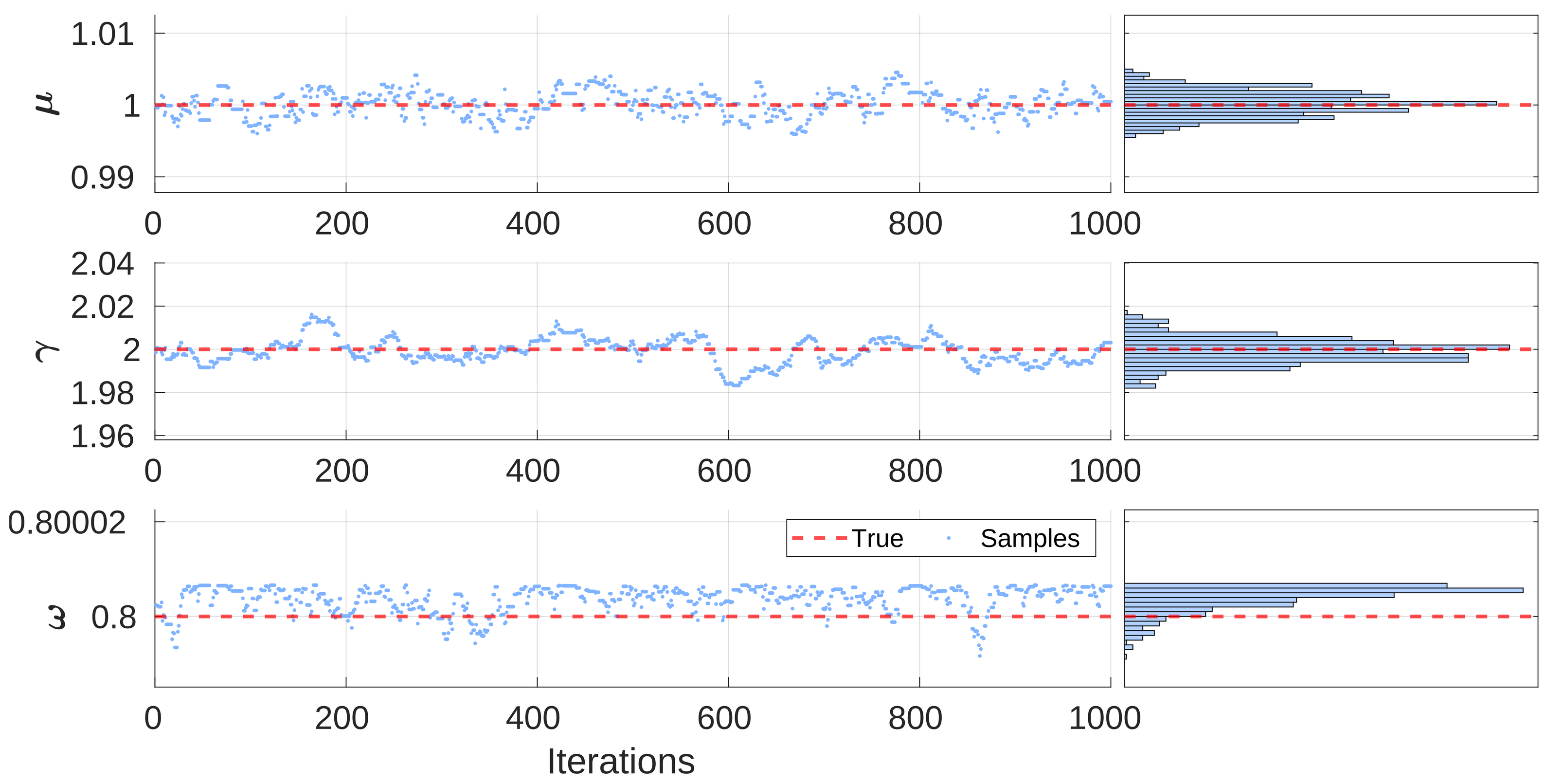
$$\pi(\boldsymbol{\eta} \mid \boldsymbol{x}) = \frac{f(\boldsymbol{x} \mid \boldsymbol{\eta})\,\pi(\boldsymbol{\eta})}{\int_H f(\boldsymbol{x}, \boldsymbol{\eta})\,\mathrm{d}\boldsymbol{\eta}} \propto f(\boldsymbol{x} \mid \boldsymbol{\eta})\,\pi(\boldsymbol{\eta})$$

with $\pi(\boldsymbol{\eta})$ the uniform and separable parameter priors. In practice, the log-posterior noted $LP(\boldsymbol{\eta}) = \log(\pi(\boldsymbol{\eta} \mid \boldsymbol{x}))$, is used.
In Metropolis-Hastings, only the target density up to a proportionality constant is needed, regardless of the proposal. When the prior is used as proposal, the acceptance ratio simplifies to a ratio of likelihoods.

Pseudo Code

```
Set η_0 to the MAP obtained by optimising π(η | x)
for t = 1 to T do
    Sample η^p from N(η_{t−1}, σI)
    Compute α = LP(η^p) − LP(η_{t−1})
    Accept with probability α: η_t = η^p, otherwise η_t = η_{t−1}
end for
```

Validation on Synthetic Data



Conclusions and Future Work

1. A positive white field ensures **pixel positivity**; correlation is introduced via filtering that preserves positivity.
   → Interpreted as a variable change yielding an **explicit partition function**.
   → Enables flexible **power spectral density modeling**.
2. Since the model's validation, several inversion strategies have been explored toward fully self-supervised deconvolution:
   → ADMM, pixel-wise optimisation, Gibbs sampling... — simple methods to assess model relevance.

Full Article